What is a unique solution in algebra

By the end of this section you should be able to: Understand the diffrence between unique solutions, no solutions, and infinitely many solutions. Reconize when a matrix has a unique solutions, no solutions, or infinitely many solutions. Reconize when a matrix has a unique solutions, no solutions, or infinitely many solutions using python. The example shown previously in this module had a unique solution. The structure of the row reduced matrix was \[\begin{split}\begin{vmatrix} 1 & 1 & -1 & | & 5 \\ 0 & 1 & -5 & | & 8 \\ 0 & 0 & 1 & | & -1 \end{vmatrix}\end{split}\] and the solution was \[x = 1\] \[y = 3\] \[z = -1\] As you can see, each variable in the matrix can have only one possible value, and this is how you know that this matrix has one unique solution Let’s suppose you have a system of linear equations that consist of: \[x + y + z = 2\] \[y - 3z = 1\] \[2x + y + 5z = 0\] The augmented matrix is \[\begin{split}\begin{vmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 1 & -3 & | & 1 \\ 2 & 1 & 5 & | & 0 \end{vmatrix}\end{split}\] and the row reduced matrix is \[\begin{split}\begin{vmatrix} 1 & 0 & 4 & | & 1 \\ 0 & 1 & -3 & | & 1 \\ 0 & 0 & 0 & | & -3 \end{vmatrix}\end{split}\] As you can see, the final row states that \[0x + 0y + 0z = -3\] which impossible, 0 cannot equal -3. Therefore this system of linear equations has no solution. Let’s use python and see what answer we get. import numpy as py from scipy.linalg import solve A = [[1, 1, 1], [0, 1, -3], [2, 1, 5]] b = [[2], [1], [0]] x = solve(A,b) x --- LinAlgError Traceback (most recent call last) in () 5 b = [[2], [1], [0]] 6 ----> 7 x = solve(A,b) 8 x C:\Users\Said Zaid-Alkailani\Anaconda3\lib\site-packages\scipy\linalg\basic.py in solve(a, b, sym_pos, lower, overwrite_a, overwrite_b, debug, check_finite, assume_a, transposed) 217 return x 218 elif 0 < info 219 raise LinAlgError('Matrix is singular.') 220 elif info > n: 221 warnings.warn('scipy.linalg.solveIll-conditioned matrix detected.' LinAlgError: Matrix is singular. As you can see the code gives us an error suggesting there is no solution to the matrix. Let’s suppose you have a system of linear equations that consist of: \[-3x - 5y + 36z = 10\] \[-x + 7z = 5\] \[x + y - 10z = -4\] The augmented matrix is \[\begin{split}\begin{vmatrix} -3 & -5 & 36 & | & 10 \\ -1 & 0 & 7 & | & 5 \\ 1 & 1 & -10 & | & -4 \end{vmatrix}\end{split}\] and the row reduced matrix is \[\begin{split}\begin{vmatrix} 1 & 0 & -7 & | & -5 \\ 0 & 2 & -3 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{vmatrix}\end{split}\] As you can see, the final row of the row reduced matrix consists of 0. This means that for any value of Z, there will be a unique solution of x and y, therefore this system of linear equations has infinite solutions. Let’s use python and see what answer we get. import numpy as py from scipy.linalg import solve A = [[-3, -5, 36], [-1, 0, 7], [1, 1, -10]] b = [[10], [5], [-4]] x = solve(A,b) x C:\Users\Said Zaid-Alkailani\Anaconda3\lib\site-packages\scipy\linalg\basic.py:223: RuntimeWarning: scipy.linalg.solve Ill-conditioned matrix detected. Result is not guaranteed to be accurate. Reciprocal condition number: 3.808655316038273e-19 ' condition number: {}'.format(rcond), RuntimeWarning) array([[-12.], [-2.], [-1.]]) As you can see we get a different type of error from this code. It states that the matrix is ill-conditioned and that there is a RuntimeWarning. This means that the computer took to long to find a unique solution so it spat out a random answer. When RuntimeWarings occur, the matrix is likely to have infinite solutions. A set of linear simultaneous equations may have (i) a unique solution, (ii) no solution, or (iii) infinitely many solutions. In a set of linear simultaneous equations, a unique solution exists if and only if, (a) the number of unknowns and the number of equations are equal, (b) all equations are consistent, and (c) there is no linear dependence between any two or more equations, that is, all equations are independent. In a system of linear simultaneous equations if one or more equations are inconsistent, the system does not have any solution. For example, if in a set of linear simultaneous equations with two equations and two unknowns, one equation is \(x+y=2\) and another equation is \(3x+3y=5\), these two equations are inconsistent within the given system. They are inconsistent because if \(x+y=2\), then \(3x+3y\) must be \(6\), not \(5\). One cannot solve the system of linear simultaneous equations \(x+y=2\) and \(3x+3y=5\) as they are inconsistent. Graphically, the solution of two linear simultaneous equations in two unknowns is equivalent to finding where the lines of the two equations cross. If these two equations are inconsistent, corresponding lines in the Cartesian plane are parallel and will never cross (See Practice Question 2). In a system of linear simultaneous equations if all equations are consistent, but (a) the number of independent equations is less than the number of unknowns, and/or (b) there exists a linear dependence between two or more equations in the system, there may exists infinitely many solutions that satisfy the system. Linear dependence, for example between two linear equations, refers to a situation when one equation in the system is a multiple of another equation. For example, equations \(y=x+2\) and \(2y=2x+4\) are linearly dependent as the later can be obtained by multiplying the former equation by \(2\). Consider a more general example. Suppose two linear simultaneous equations are: $$a_{11}x_{1} + a_{12}x_{2} = b_{1}$$ $$a_{21}x_{1} + a_{22}x_{2} = b_{2}$$ Where, \(a_{ij}\) is the coefficient, \(x_j\) is the variable, and \(b_i\) is the constant. In this system if \(a_{1j}=ka_{2j}\) and \(b_1=kb_2\), where \(k\) is a constant, the equations are linearly dependent. Graphically the lines representing the graphs of two equations coincide if the equations are linearly dependent, and every point on either line is a solution. [See Practice Question 3] One interesting form of linear dependence may arise in a system of \(m×n\) linear simultaneous equations when one equation is the sum or difference of more than one equation in the system. For example, equations (i) \(x+y+z=10\),(ii) \(2x-2y-2z=4\), and (iii) \(3x-y-z=14\) have linear dependence. (Why?) To sum up, consider a system of linear simultaneous equations where all equations are consistent, however, due to the linear dependence between some equations the number of independent equations is less than the number of unknowns. Such a system has infinitely many solutions. It follows from the discussion in this section is that two linear simultaneous equations in two unknowns can have a unique solution, no solution or infinitely many solutions and this is true for every system of linear simultaneous equations with \(m\) equations and \(n\) unknowns. To read more about the existence of a unique solution, inconsistency, and linear dependence, please see the recommended books. Something went wrong. Wait a moment and try again. $\begingroup$ General definition of System of Linear Equations says that "If The system has a unique solution, It has independent set of Equations" Consider the system of linear equations $$x-2y=-1$$ $$3x+5y=8$$ $$4x+3y=7$$ As we can see from the below graph that all the 3 line intersect at a single point $\implies$ System has a unique solution. But at the same time system is not independent as any equation can be derived from the algebraic manipulations of other two equations. So, how definition is true. $\endgroup$ 5 The GMAT sample question in quant given below is a Linear Equations question and tests concepts related to types of solutions for a system of linear equations. This concept is usually tested in the GMAT as a data sufficiency question rather than as a problem solving question. A sub 600 level GMAT practice question in system of linear equations. Question 3: For what values of 'k' will the pair of equations 3x + 4y = 12 and kx + 12y = 30 NOT have a unique solution? Get to Q51 in GMAT QuantOnline GMAT Course @ INR 3500 GMAT Live Online ClassesStarts Sat, Apr 23, 2022 A system of linear equations ax + by + c = 0 and dx + ey + g = 0 will have a unique solution if the two lines represented by the equations ax + by + c = 0 and dx + ey + g = 0 intersect at a point. i.e., if the two lines are neither parallel nor coincident. Essentially, the slopes of the two lines should be different. What does that translate into? ax + by + c = 0 and dx + ey + g = 0 will intersect at one point if their slopes are different. Express both the equations in the standardized y = mx + c format, where 'm' is the slope of the line and 'c' is the y-intercept. ax + by + c = 0 can be written as y = \-\frac{a}{b}x -\frac{c}{a}) And dx + ey + g = 0 can be written as y = \-\frac{d}{e}x -\frac{g}{e}) Slope of the first line is \-\frac{a}{b}) and that of the second line is \-\frac{d}{e}) For a unique solution, the slopes of the lines should be different. ∴ \-\frac{a}{b} eq -\frac{d}{e}) Or \\frac{a}{d} eq \frac{b}{e}) Condition for the equations to NOT have a unique solution The slopes should be equal Or \\frac{a}{d} = \frac{b}{e}) Apply the condition in the given equations to find k In the question given above, a = 3, b = 4, d = k and e = 12. Therefore, \\frac{3}{k} = \frac{4}{12}) Or 'k' should be equal to 9 for the system of linear equations to NOT have a unique solution. The question is "What is the value of k? When k = 9, the system of equations will represent a pair of parallel lines (their y-intercepts are different). So, there will be NO solution to this system of linear equations in two variables. Choice A is the correct answer.